# Periodic solutions of hybrid jump diffusion processes

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# Outline

1. Periodic solutions of stochastic dynamical systems
2. Hybrid jump diffusion processes
3. Strong Feller property and irreducibility
   - SDE case
   - SPDE case
4. Examples

# Outline

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Periodic solutions: key concept in the theory of dynamical systems.

Poincaré (1881): pioneering work.

Khasminskii: stochastic stability of differential equations.

Markov process $\{X(t), t \geq 0\}$ is $\theta$-periodic if for any $n \in \mathbb{N}$ and any $0 \leq t_1 < t_2 < \cdots < t_n$, the joint distribution of $X(t_1 + k\theta), X(t_2 + k\theta), \ldots, X(t_n + k\theta)$ is independent of $k$ for $k \in \mathbb{N} \cup \{0\}$.

Markovian transition semigroup $\{P_{s,t}\}$ is $\theta$-periodic if $P(s, x, t, A) = P(s + \theta, x, t + \theta, A)$ for any $0 \leq s < t$, $x \in E$ and $A \in \mathcal{B}(E)$.

Probability measures $\{\mu_s, s \geq 0\}$ on $(E, \mathcal{B}(E))$ is $\theta$-periodic w.r.t. $\{P_{s,t}\}$ if

$$\mu_s(A) = \int_E P(s, x, s+\theta, A)\mu_s(dx), \quad \forall A \in \mathcal{B}(E), \; s \geq 0.$$

Stochastic process $\{X(t), t \geq 0\}$ with values in $\mathbb{R}^d$ or $H$ is a $\theta$-periodic solution of the SDE/SPDE

$$\begin{aligned} dX(t) &= b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t) + \int_{\{|u|<1\}} H(t, X(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ &\quad + \int_{\{|u|\geq 1\}} G(t, X(t-), u)N(\mathrm{d}t, \mathrm{d}u) \end{aligned}$$

if it is a solution and is $\theta$-periodic.

## Existence of periodic solutions

Krylov-Bogoliubov There exists a $\theta$-periodic Markov process with a given $\theta$-periodic transition prob. funct. $P(s,x,t,A)$ if for some probability measure $\nu$ on $(E,\mathcal{B}(E))$,

$$\frac{1}{T}\int_0^T\int_E P(s,x,s+u,\cdot)\nu(dx)du \text{ is tight.}$$

$$E=\mathbb{R}^d:\lim_{R\to\infty}\liminf_{T\to\infty}\frac{1}{T}\int_0^T P(s,x,s+u,B_R^c)du=0.$$

## Uniqueness of periodic solutions

Doob Let $0 \leq s < t < t_1$. If $\{P_{s,t}\}$ is strong Feller at $(t, t_1)$ and irreducible at $(s, t)$, then it is regular at $(s, t_1)$.

If $\{P_{s,t}\}$ is regular at $(s, s+\theta)$ for any $s \in [0, \theta)$, then there exists at most one $\theta$-periodic measure.

# Other definitions of periodic solutions

Deterministic dynamical system $\Psi : E \to E$.

A periodic solution with period $\theta > 0$ is a function $\psi : \mathbb{R} \to E$ such that

$$\Psi_t(\psi(s)) = \psi(t+s), \quad \psi(t+\theta) = \psi(t), \quad \forall s, t \in \mathbb{R}.$$

Random dynamical system $\Psi : E \to E$ over probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

$\theta_t : \Omega \to \Omega$: measurable $P$-measure preserving map.

$$\Psi(0,\omega)x = x, \ \Psi(t+s,\omega) = \Psi(t,\theta_s\omega)\Psi(s,\omega), \ \ \forall x \in E, \ s,t \in \mathbb{R}.$$

Zhao and Zheng (09) A random periodic solution with period $\Theta > 0$ is an $\mathcal{F}$-measurable function $\psi : \Omega \times \mathbb{R} \to E$ such that

$$\Psi(t,\omega)\psi(s,\omega) = \psi(t+s,\theta_t\omega), \ \ \psi(s+\Theta,\omega) = \psi(s,\omega), \ \ \forall s,t \in \mathbb{R}.$$

Sun and Zheng (20) A weak random periodic solution of $\Psi$ with period $\Theta$ is a pair of measurable maps $\psi : \mathbb{R} \times \Omega \to E$ and $\Theta : \Omega \to (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$\Psi(t,\omega)\psi(s,\omega) = \psi(t+s, \theta_t\omega), \;\; \psi(s+\Theta(\theta_{-s}\omega), \omega) = \psi(s,\omega), \;\; \forall s,t \in \mathbb{R}.$$

# Outline

## Regime-switching jump diffusion

$E$: $\mathbb{R}^d$ or $H$.

$(X(t), \Lambda(t))$: Markov process on $E \times \mathbb{N}$.

$$
\begin{aligned}
dX(t) &= b(t, X(t), \Lambda(t))\mathrm{d}t + \sigma(t, X(t), \Lambda(t))\mathrm{d}B(t) \\
&\quad + \int_{\{|u|<1\}} H(t, X(t-), \Lambda(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\
&\quad + \int_{\{|u|\geq 1\}} G(t, X(t-), \Lambda(t-), u)N(\mathrm{d}t, \mathrm{d}u).
\end{aligned}
$$

$$
\mathbb{P}\{\Lambda(t+\Delta) = j | \Lambda(t) = i, X(t) = x\} = \begin{cases} q_{ij}(x)\Delta + o(\Delta), & i \neq j, \\ 1 + q_{ij}(x)\Delta + o(\Delta), & i = j. \end{cases}
$$

$$L = \sup_{x \in E,\, i \in \mathbb{N}} \sum_{j \neq i} q_{ij}(x) < \infty.$$

$$\Delta_{i1}(x) = [0, q_{i1}(x)), \quad \Delta_{i2}(x) = [q_{i1}(x), q_{i1}(x) + q_{i2}(x)), \cdots,$$
$$\Delta_{ij}(x) = \left[\sum_{s=1}^{j-1} q_{is}(x), \sum_{s=1}^{j} q_{is}(x)\right), \cdots.$$

$$h(x,i,r) = \begin{cases} j-i, & \text{if } r \in \Delta_{ij}(x), \\ 0, & \text{otherwise.} \end{cases}$$

$$d\Lambda(t) = \int_{[0,L]} h(X(t-), \Lambda(t-), r) N_1(dt, dr).$$

$$\mathcal{A}f(t,x,i) := \mathcal{L}_i f(t,x,i) + Q(x)f(t,x,\cdot)(i).$$

$$\begin{aligned}
&\mathcal{L}_i f(\cdot,\cdot,i)(t,x)\\
= \ & f_t(t,x,i) + \langle f_x(t,x,i), b(t,x,i)\rangle + \frac{1}{2}\mathrm{trace}(\sigma^*(t,x,i) f_{xx}(t,x,i)\sigma(t,x,i))\\
&+ \int_{\{|u|<1\}} [f(t,x+H(t,x,i,u),i) - f(t,x,i) - \langle f_x(t,x,i), H(t,x,i,u)\rangle]\nu(\mathrm{d}u)\\
&+ \int_{\{|u|\geq 1\}} [f(t,x+G(t,x,u,i),i) - f(t,x,i)]\nu(\mathrm{d}u),
\end{aligned}$$

and

$$Q(x)f(t,x,\cdot)(i) = \sum_{j\in\mathbb{N}} q_{ij}(x)[f(t,x,j) - f(t,x,i)].$$

## Existence and uniqueness of solutions

$$b(t+\theta,x,i)=b(t,x,i),\quad \sigma(t+\theta,x,i)=\sigma(t,x,i),$$
$$H(t+\theta,x,i,u)=H(t,x,i,u),\quad G(t+\theta,x,i,u)=G(t,x,i,u).$$

There exists a positive increasing function $f$ on $\mathbb{N}$ satisfying $\lim_{j\to\infty}f(j)=\infty$ and

$$\sup_{x\in E,\, i\in\mathbb{N}}\sum_{j\neq i}[f(j)-f(i)]q_{ij}(x)<\infty.$$

## SDE case

**A1)** For each $i \in \mathbb{N}$,

$$b(\cdot, 0, i)\ , \sigma(\cdot, 0, i) \in L^2([0, \theta); \mathbb{R}^d), \ \ \int_{\{|u|<1\}} |H(\cdot, 0, i, u)|^2 \nu(\mathrm{d}u) \in L^1([0, \theta); \mathbb{R}^d).$$

For each $n \in \mathbb{N}$, there exists $L_n(t) \in L^1([0, \theta); \mathbb{R}_+)$ such that for any $t \in [0, \theta)$, $i \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq n$,

$$\begin{aligned} &|b(t, x, i) - b(t, y, i)|^2 \leq L_n(t)|x - y|^2, \\ &|\sigma(t, x, i) - \sigma(t, y, i)|^2 \leq L_n(t)|x - y|^2, \\ &\int_{\{|u|<1\}} |H(t, x, i, u) - H(t, y, i, u)|^2 \nu(\mathrm{d}u) \leq L_n(t)|x - y|^2. \end{aligned}$$

**A2)** There exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$ and $g \in L^1_{loc}([0,\infty); \mathbb{R})$ such that

$$\lim_{|x|\to\infty} \left[ \inf_{t\in[0,\infty)} V(t,x) \right] = \infty,$$

and for any $t \geq 0$, $i \in \mathbb{N}$ and $x \in \mathbb{R}^d$

$$\mathcal{L}_i V(t,x) \leq g(t).$$

**Theorem 1**

*Suppose that assumptions* $\mathbf{A1}$ *and* $\mathbf{A2}$ *hold. Then, the hybrid system has a unique strong solution* $(X(t), \Lambda(t))$.

# SPDE case (variational approach)

$H$, $U$: Hilbert space.

$V$: reflexive Banach space continuously and densely embedded into $H$.

$B(t)$: cylindrical $U$-valued Wiener process.

$N(dt,dz)$: independent Poisson random measure on Banach space $Z$.

$$
\begin{aligned}
dX(t) &= b(t,X(t))\mathrm{d}t + \sigma(t,X(t))\mathrm{d}B(t) + \int_{\{|u|<1\}} H(t,X(t),u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u) \\
&\quad + \int_{\{|u|\geq 1\}} G(t,X(t),u)N(\mathrm{d}t,\mathrm{d}u).
\end{aligned}
$$

Suppose there exist $\alpha > 1$, $\beta \geq 0$, $\vartheta > 0$, $K > 0$, $\gamma < \frac{\vartheta}{2\beta}$, $c > 0$, $C \in L_{loc}^{\frac{\beta+2}{2}}([0,\theta);\mathbb{R}_+)$ and $\rho \in L_{loc}^{\infty}(V;\mathbb{R}_+)$ such that for all $v_1, v_2, v \in V$ and $t \in [0,\theta)$:

(Hemicontinuity) $s \mapsto_{V^*} \langle b(t, v_1 + s v_2), v \rangle_V$ is continuous on $\mathbb{R}$.

(Local Monotonicity)

$$
\begin{aligned}
& 2_{V^*}\langle b(t,v_1) - b(t,v_2), v_1 - v_2 \rangle_V + \|\sigma(t,v_1) - \sigma(t,v_2)\|_{L_2(H)}^2 \\
& + \int_{\{|z|<1\}} |H(t,v_1,z) - H(t,v_2,z)|_H^2 \nu(dz) \\
\leq \quad & (K + \rho(v_2))|v_1 - v_2|_H^2.
\end{aligned}
$$

(Coercivity)

$$2_{V^*}\langle b(t,v),v\rangle_V + \|\sigma(t,v)\|^2_{L_2(H)} + \int_{\{|z|<1\}} |H(t,v,z)|^2_H \nu(dz)$$
$$\leq \quad C(t) - \vartheta |v|^\alpha_V + c|v|^2_H.$$

(Growth of $b$)

$$|b(t,v)|^{\frac{\alpha}{\alpha-1}}_{V^*} \leq [C(t) + c|v|^\alpha_V](1 + |v|^\beta_H).$$

(Growth of $\sigma$ and $H$)

$$\|\sigma(t,v)\|^2_{L_2(H)} + \int_{\{|z|<1\}} |H(t,v,z)|^2_H \nu(dz) \leq C(t) + \gamma|v|^\alpha_V + c|v|^2_H.$$

(Growth of $H$ in $L^{\beta+2}$)

$$\int_{\{|z|<1\}} |H(t,v,z)|_H^{\beta+2}\nu(dz) \leq [C(t)]^{\frac{\beta+2}{2}} + c|v|_H^{\beta+2}.$$

(Growth of $\rho$)

$$\rho(v) \leq c(1+|v|_V^{\alpha})(1+|v|_H^{\beta}).$$

Brzeźniak, Liu and Zhu (14) Let $T > 0$. For any $x \in L^{\beta+2}((\Omega, \mathcal{F}_0, \mathbb{P}); H)$, there exists a unique $H$-valued adapted cadlag process $\{X(t)\}_{t\in[0,T]}$ such that

Its $dt \times \mathbb{P}$-equivalent class $\widehat{X}$ is in $L^\alpha([0,T]; V) \cap L^2([0,T]; H)$ $\mathbb{P}$-a.s..

For any progressively measurable $dt \times \mathbb{P}$-version $\overline{X}$ of $\widehat{X}$, the following equality holds $\mathbb{P}$-a.s.:

$$\begin{aligned} X(t) &= x + \int_0^t b(s, \overline{X}(s))dt + \int_0^t \sigma(s, \overline{X}(s))dB(s) \\ &+ \int_0^t \int_{\{|z|<1\}} H(s, \overline{X}(s), z)\widetilde{N}(ds, dz) \\ &+ \int_0^t \int_{\{|z|\geq 1\}} G(s, \overline{X}(s), z)N(ds, dz). \end{aligned}$$

# Outline

Kill the process $X^{(i)}(t)$ with rate $q_i(x)=\sum_{j\neq i}q_{ij}(x)$ and obtain a subprocess $\widetilde{X}^{(i)}(t)$ with generator $\mathcal{L}-q_i$.

Let $\widetilde{P}^{(i)}(s,x,\cdot)$ be the transition probability function of $\widetilde{X}^{(i)}(t)$. Then, for $0\le s<t$, $B\in\mathcal{B}(E)$ and $j\in\mathbb{N}$,

$$
\begin{aligned}
& P(s,(x,i),t,B\times\{j\}) \\
= \ & \delta_{ij}\widetilde{P}^{(i)}(s,x,t,B) \\
& +\int_s^t\sum_{j_1\in\mathbb{N}\setminus\{i\}}\int_E P(t_1,(x_1,j_1),t,B\times\{j\})q_{ij_1}(x_1)\widetilde{P}^{(i)}(s,x,t_1,dx_1)dt_1.
\end{aligned}
$$

# SDE case

**(A3)** (i) $b(\cdot,0)\in L^2([0,\theta);\mathbb{R}^d)$, $\sigma(\cdot,0)\in L^\infty([0,\theta);\mathbb{R}^d)$, $\int_{\{|u|<1\}}|H(\cdot,0,u)|^2\nu(\mathrm{d}u)\in L^1([0,\theta);\mathbb{R}^d)$.

(ii) For each $n\in\mathbb{N}$, there exists $L_n\in L^\infty([0,\theta);\mathbb{R}_+)$ such that for any $t\in[0,\theta)$ and $x,y\in\mathbb{R}^d$ with $|x|\vee|y|\le n$,

$$|b(t,x)-b(t,y)|^2\le L_n(t)|x-y|^2,\quad |\sigma(t,x)-\sigma(t,y)|^2\le L_n(t)|x-y|^2,$$
$$\int_{\{|u|<1\}}|H(t,x,u)-H(t,y,u)|^2\nu(\mathrm{d}u)\le L_n(t)|x-y|^2.$$

**(A4)** For any $t\in[0,\theta)$ and $x\in\mathbb{R}^d$, $Q(t,x):=\sigma(t,x)\sigma^T(t,x)$ is invertible and

$$\sup_{|x|\le n,\, t\in[0,\theta)}|Q^{-1}(t,x)|<\infty,\quad \forall n\in\mathbb{N}.$$

**Lemma** Suppose that $(\mathbf{A3})$ and $(\mathbf{A4})$ hold. Let $T > 0$. Then, there exists a constant $M_T > 0$ such that for all $\varphi \in B_b(\mathbb{R}^d)$ and $0 \le s < t \le T$,

$$|\mathbb{E}_{s,x}[\varphi(X_n(t))] - \mathbb{E}_{s,y}[\varphi(X_n(t))]| \le \frac{M_T}{\sqrt{t-s}} \|\varphi\|_\infty |x-y|, \quad \forall x, y \in \mathbb{R}^d.$$

**$(\mathbf{H_w})$** There exists $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$ such that

$$\lim_{|x|\to\infty} \left[ \inf_{t\in[0,\infty)} V(t,x) \right] = \infty,$$

and

$$\sup_{x\in\mathbb{R}^d,\, t\in[0,\infty)} \mathcal{L}V(t,x) < \infty.$$

**Theorem 2**

*Suppose that $(\mathbf{A3})$, $(\mathbf{A4})$ and $(\mathbf{H_w})$ hold. Then $\{P_{s,t}\}$ is strong Feller.*

Let $f$ be a function on $[0,\infty)\times\mathbb{R}^d$. For $\rho>0$, define

$$f^{\bullet\rho}(t,x)=f(t,\rho x),\quad t\geq 0, x\in\mathbb{R}^d.$$

**(H)** There exists $V\in C^{1,2}([0,\infty)\times\mathbb{R}^d;\mathbb{R}_+)$ satisfying the following conditions:
(i)

$$\lim_{|x|\to\infty}\left[\inf_{t\in[0,\infty)}V(t,x)\right]=\infty.$$

(ii) For any $\rho \geq 1$, there exist $q_\rho \in B_{b,loc}(\mathbb{R}_+)$ and $W_\rho(t,x) \in B_{b,loc}([0,\infty) \times \mathbb{R}^d; \mathbb{R}^d)$ satisfying for each $n \in \mathbb{N}$ there exists $R_n \in L^1_{loc}([0,\infty); \mathbb{R}_+)$ such that for any $t \in [0,\infty)$ and $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq n$,

$$|W_\rho(t,x) - W_\rho(t,y)|^2 \leq R_n(t)|x-y|^2,$$

and for $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\mathcal{L}V^{\bullet\rho}(t,x) \leq q_\rho(t),$$

$$V^{\bullet\rho}(t,x) \leq \langle W_\rho, \nabla_x V^{\bullet\rho}(t,x) \rangle.$$

**Theorem 3**

*Suppose that* $(\mathbf{A4})$ *and* $(\mathbf{H})$ *hold. Then* $\{P_{s,t}\}$ *is irreducible.*

Sketch of proof: Let $x, y \in \mathbb{R}^d$ with $x \neq y$ and $T > 0$. For $r \in \mathbb{N}$, we consider the following SDE:

$$dX^r(t) = [b(t, X^r(t)) - rW(t, X^r(t) - y)]\, dt + \sigma(t, X^r(t))dB(t) + \int_{\{|u|<1\}} H(t, X^r(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) + \int_{\{|u|\geq 1\}} G(t, X^r(t-), u)N(\mathrm{d}t, \mathrm{d}u).$$

The generator $\mathcal{L}^W$ is given by

$$\mathcal{L}^{W_\rho} f(t, x) = \mathcal{L}f(t, x) - r\langle W_\rho, \nabla_x f\rangle(t, x).$$

$$
\begin{aligned}
& \mathbb{E}[e^{rt}V_3^{\bullet\rho}(t,X^r(t)-y)] \\
= \; & \mathbb{E}[V_3^{\bullet\rho}(0,x-y)] + \mathbb{E}\left[\int_0^t e^{rv}\mathcal{L}^{W_\rho}V_3^{\bullet\rho}(v,X^r(v)-y)\mathrm{d}v\right] \\
& +\mathbb{E}\left[\int_0^t re^{rv}V_3^{\bullet\rho}(s,X^r(v)-y)\mathrm{d}v\right] \\
\leq \; & \mathbb{E}[V_3^{\bullet\rho}(0,x-y)] + \mathbb{E}\left[\int_0^t e^{rv}[-rV_3^{\bullet\rho}(v,X^r(v)-y)+q_\rho(v)]\mathrm{d}v\right] \\
& +\mathbb{E}\left[\int_0^t re^{rv}V_3^{\bullet\rho}(v,X^r(v)-y)\mathrm{d}v\right] \\
= \; & \mathbb{E}[V_3^{\bullet\rho}(0,x-y))] + \int_0^t q_\rho(v)e^{rv}\mathrm{d}v.
\end{aligned}
$$

Hence

$$
\mathbb{E}[V_3^{\bullet\rho}(t,X^r(t)-y)] \leq \frac{\mathbb{E}[V_3^{\bullet\rho}(0,x-y)]}{e^{rt}} + \frac{(1-e^{-rt})\sup_{0\leq v\leq T}|q_\rho(v)|}{r}.
$$

Then, for any $0 < a < |x-y|$, there exists $r_a \in \mathbb{N}$ such that

$$\begin{aligned}
& \mathbb{P}(|X^{r_a}(T) - y| \geq a) \\
\leq\ & \mathbb{P}\{V_3^{\bullet\rho_a}(T, X^{r_a}(T) - y) \geq 1\} \\
\leq\ & \mathbb{E}[V_3^{\bullet\rho_a}(T, X^{r_a}(T) - y)] \\
\leq\ & \frac{\mathbb{E}[V_3^{\bullet\rho_a}(0, x-y)]}{e^{r_a T}} + \frac{(1 - e^{-r_a T}) \sup_{0 \leq v \leq T} |q_{\rho_a}(v)|}{r_a} \\
<\ & \frac{1}{2}.
\end{aligned}$$

Define $\tau_K := \inf\{t : |X^{r_a}(t)| \geq K\}$. There exists $K \in \mathbb{N}$ such that

$$\mathbb{P}(\tau_K \leq T) < \frac{1}{2}.$$

Define

$$\alpha(t) := -r_a(\sigma(t, X^{r_a}(t)))^T[\sigma(t, X^{r_a}(t))(\sigma(t, X^{r_a}(t)))^*]^{-1}W_{\rho_a}(t, X^{r_a}(t)-y),$$

$$\widetilde{B}(t) := B(t) + \int_0^{t\wedge\tau_K} \alpha(v)dv,$$

and

$$R_t = \exp\left(\int_0^{t\wedge\tau_K} \alpha(v)dB(v) - \frac{1}{2}\int_0^{t\wedge\tau_K} |\alpha(v)|^2 dv\right).$$

By Girsanov's theorem, under the new probability measure $d\mathbb{Q} = \mathbb{R}_T d\mathbb{P}$, $\widetilde{B}(t)$ is still a Brownian motion, and $N(dt, du)$ is a Poisson random measure with the same compensator $\nu(du)dt$.

# SPDE case

**(N)** There exist positive self-adjoint operators $\{\sigma_n\} \subset L_2(H)$ such that for all $n \in \mathbb{N}$, $t \geq 0$, $v \in V$ with $|v|_H \leq n$,

$$\sigma(t,v)[\sigma(t,v)]^* \geq \sigma_n^2.$$

**(Lip)** For $n \in \mathbb{N}$, there exists $C_n > 0$ independent of $t$ such that for all $v_1, v_2 \in V$ with $|v_1|_H, |v_2|_H \leq n$,

$$\|\sigma(t,v_1) - \sigma(t,v_2)\|_{L_2(H)} \leq C_n |v_1 - v_2|_H.$$

**(D)** There exist $\lambda \in [2, \infty) \cap (\alpha - 2, \infty)$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, there exist $\widetilde{K_n} \geq 0$ and $\delta_n > 0$, which are independent of $t$, such that for all $v_1, v_2 \in V$ and $t \geq 0$,

$$2_{V^*}\langle b(t, v_1) - b(t, v_2), v_1 - v_2 \rangle_V + \|\sigma(t, v_1) - \sigma(t, v_2)\|^2_{L_2(H)}$$
$$+ \int_{\{|z|<1\}} |H(t, v_1, z) - H(t, v_2, z)|^2_H \nu(dz)$$
$$\leq -\delta_n |\sigma_n{}^{-1}(v_1 - v_2)|^\lambda_H |v_1 - v_2|^{\alpha-\lambda}_H + \widetilde{K_n} |v_1 - v_2|^2_H.$$

**Theorem 4**

*Suppose that* $(\mathbf{N})$, $(\mathbf{Lip})$ *and* $(\mathbf{D})$ *hold. Then* $\{P_{s,t}\}$ *is strong Feller.*

Sketch of proof: Following the papers of F.Y. Wang, W. Liu and S.Q. Zhang, we consider coupling equations

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) + \int_{\{|z|<1\}} H(t, X(t), z)\widetilde{N}(dt, dz),$$

$$dY(t) = b(t, Y(t))dt + \sigma(t, Y(t))dB(t) + \int_{\{|z|<1\}} H(t, Y(t), z)\widetilde{N}(dt, dz)$$

$$-|x-y|_H^{\alpha'} \frac{X(t) - Y(t)}{|X(t) - Y(t)|_H^{\varepsilon}} dt.$$

$$\tau := \inf\{t > 0 : |X(t) - Y(t)|_H = 0\}.$$

Define

$$\widetilde{B}(t) = B(t) + \int_0^{t\wedge\tau} |x-y|_H^{\alpha'} \frac{[\sigma(s,Y(s))]^*(\sigma(s,Y(s))[\sigma(s,Y(s))]^*)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}} ds,$$

$$R_t = \exp\left\{ |x-y|_H^{\alpha'} \int_0^{t\wedge\tau} \frac{[\sigma(s,Y(s))]^*(\sigma(s,Y(s))[\sigma(s,Y(s))]^*)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}} dB(s) - \frac{|x-y|_H^{2\alpha'}}{2} \int_0^{t\wedge\tau} \frac{|[\sigma(s,Y(s))]^*(\sigma(s,Y(s))[\sigma(s,Y(s))]^*)^{-1}(X(s)-Y(s))|_H^2}{|X(s)-Y(s)|_H^{2\varepsilon}} ds \right\}.$$

Through moment estimation, we can show that $\{\widetilde{B}(t)\}_{t\in[0,T]}$ is cylindrical Wiener process on $H$ under prob. measure $d\mathbb{Q} = R_T d\mathbb{P}$.

$$dY(t) = b(t,Y(t))dt + \sigma(t,Y(t))d\widetilde{B}(t) + \int_{\{|z|<1\}} H(t,Y(t),z)\widetilde{N}(dt,dz).$$

$$
\begin{aligned}
& |P_{0,T}f(x) - P_{0,T}f(y)| \\
= \; & |\mathbb{E}[f(X_T)] - \mathbb{E}[f(R_T f(Y_T))]| \\
= \; & |\mathbb{E}[f(Y_T)(1 - R_T)1_{\{\tau<T\}}]| + |\mathbb{E}[(f(X_T) - f(Y_T))1_{\{\tau\geq T\}}| \\
\leq \; & |f|_\infty \{\mathbb{E}[|1 - R_T|] + \mathbb{P}(\tau \geq T)\}.
\end{aligned}
$$

Estimate upper bounds of $\mathbb{E}[|1 - R_T|]$ and $\mathbb{P}(\tau \geq T)$.

$P_{s,t}f$ is Hölder continuous on $H$ for all $f \in \mathcal{B}_b(H)$.

**Theorem 5**

*Suppose that* $(\mathbf{N})$ *holds and* $\overline{\{u \in H : b(t,u) \in H, \forall t \in [0,T]\}} = H$. *Then* $\{P_{s,t}\}$ *is irreducible.*

# Outline

## Example 1 (Stochastic Lorenz equation with regime switching)

$$
\begin{aligned}
dX_1(t) &= [-\alpha(t,\Lambda(t))X_1(t) + \alpha(t,\Lambda(t))X_2(t)]dt + \sum_{j=1}^{3}\sigma_{1j}(t,X(t),\Lambda(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_1(t,X(t-),\Lambda(t-),u)\widetilde{N}(dt,du) \\
&\quad + \int_{\{|u|\geq 1\}} G_1(t,X(t-),\Lambda(t-),u)N(dt,du), \\
dX_2(t) &= [\mu(t,\Lambda(t))X_1(t) - X_2(t) - X_1(t)X_3(t)]dt + \sum_{j=1}^{3}\sigma_{2j}(t,X(t),\Lambda(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_2(t,X(t-),\Lambda(t-),u)\widetilde{N}(dt,du) \\
&\quad + \int_{\{|u|\geq 1\}} G_2(t,X(t-),\Lambda(t-),u)N(dt,du), \\
dX_3(t) &= [-\beta(t,\Lambda(t))X_3(t) + X_1(t)X_2(t)]dt + \sum_{j=1}^{3}\sigma_{3j}(t,X(t),\Lambda(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_3(t,X(t-),\Lambda(t-),u)\widetilde{N}(dt,du) \\
&\quad + \int_{\{|u|\geq 1\}} G_3(t,X(t-),\Lambda(t-),u)N(dt,du).
\end{aligned}
$$

$$q_{ij}(x)\begin{cases} = 0, & |i-j| > k, \\ \in (0, M], & 0 < |i-j| \le k, \end{cases}$$

$$\inf_{x\in\mathbb{R}^3,\, i>k,\, i-k\le j<i}\{q_{ij}(x)\} > \sup_{x\in\mathbb{R}^3,\, i>k,\, i<j\le i+k}\{q_{ij}(x)\}.$$

For $i \in \mathbb{N}$, $t \in [0,\theta)$, $x \in \mathbb{R}^3$, $Q(t,x) = \sigma(t,x)\sigma^T(t,x)$ is invertible and

$$\sup_{|x|\le n,\, t\in[0,\theta)} |Q^{-1}(t,x,i)| < \infty, \quad \forall n \in \mathbb{N}, i \in \mathbb{N}.$$

For any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for any $i \in \mathbb{N}$,

$$|\sigma(t,x,i)|^2 + \int_{\{|u|<1\}} |H(t,x,i,u)|^2\nu(du) + \int_{\{|u|\ge 1\}} |G(s,x,i,u)|^2\nu(du)$$
$$\le \varepsilon|x|^2 + c_\varepsilon.$$

Example 2 (Stochastic porous media equation with regime switching)

$D \subset \mathbb{R}^d$: bounded domain with smooth boundary.

$$L = -(-\Delta_D)^\gamma, \quad \gamma > d/2.$$

$$V = L^{r+1}(D; dx) \subset H = H^\gamma(D; dx) \subset V^* = (L^{r+1}(D; dx))^*, \quad r > 1.$$

$$\Psi(s) = s|s|^{r-1}, \quad \Phi(s) = cs, \quad s \in \mathbb{R},\ c \geq 0.$$

$$\begin{aligned} dX(t) &= [L\Psi(X(t)) + \Phi(X(t))]\,\mathrm{d}t + \sigma(t, X(t), \Lambda(t))\mathrm{d}B(t) \\ &\quad + \int_{\{|u|<1\}} H(t, X(t), \Lambda(t), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ &\quad + \int_{\{|u|\geq 1\}} G(t, X(t), \Lambda(t), u)N(\mathrm{d}t, \mathrm{d}u). \end{aligned}$$

Let $\lambda_1 \le \lambda_2 \le \cdots$ be eigenvalues of $-\Delta_D$ and $\{e_i\}$ be corresponding unit eigenfunctions.

$$|\sigma_j(t,x,i) - \sigma_j(t,y,i)| \le c|x-y|, \quad \forall x,y \in H^\gamma(D;dx), t \in [0,\theta), i \in \mathbb{N},$$

$$\inf_{|x|\le R,\, t\in[0,\theta),\, i,j\in\mathbb{N}} \sigma_j(t,x,i) > 0, \qquad \forall R > 0,$$

$$\sigma(t,x,i)e_j = \sigma_j(t,x,i)j^{-\gamma/d}e_j, \quad j \ge 1.$$

$$0 < \inf_{x\in H^\gamma(D;dx),\, j\neq i} \{j^{1+\delta} q_{ij}(x)\} < \sup_{x\in H^\gamma(D;dx),\, j\neq i} \{j^{1+\delta} q_{ij}(x)\} < \infty, \quad \delta > 0.$$

For any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for any $i \in \mathbb{N}$,

$$\begin{aligned} & \int_{\{|u|<1\}} |H(t,x,i,u)|^2 \nu(du) + \int_{\{|u|\geq 1\}} |G(s,x,i,u)|^2 \nu(du) \\ \leq \quad & \varepsilon |x|^2 + c_\varepsilon . \end{aligned}$$

The hybrid system has a unique $\theta$-periodic solution $(X(t), \Lambda(t))$.

$\{P_{s,t}\}$ is strong Feller and irreducible.

Let $\mu_s(A) = \mathbb{P}\{(X(s), \Lambda(s)) \in A\}$. Then, for any $s \geq 0$ and $\varphi \in L^2(E \times \mathbb{N}; \mu_s)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P_{s,s+i\theta}\varphi = \int_{E\times\mathbb{N}} \varphi d\mu_s \ \text{ in } \ L^2(E \times \mathbb{N}; \mu_s).$$

## Some references

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